Math 246C Lecture 8 Notes

Daniel Raban

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1 Inversion of the Fourier-Laplace Transform

1.1 Bounds on analytic functionals

Last time, we were talking about analytic functionals $\mu : \mathrm{Hol}(\mathbb{C}) \to \mathbb{C}$. We defined the Fourier-Laplace transform $\widehat{\mu}(\zeta) = \mu_z(e^{z\zeta}), z \in \mathbb{C}$. Assume that μ is carried by the compact set $K \subseteq \mathbb{C}$: for all neighborhoods ω of K,

$$|\mu(f)| \leq C_\omega \sup_\omega |f|, \qquad f \in \mathrm{Hol}(\mathbb{C}).$$

So there exists a measure ν on $\overline{\omega}$ such that

$$\mu(f) = \int_{\overline{\omega}} f(z)\, d\nu(z).$$

So we get the bound

$$|\widehat{\mu}(\zeta)| \leq \exp(\sup_{z \in \overline{\omega}} \mathrm{Re}(z\zeta)) \int_{\overline{\omega}} |d\nu(z)|.$$

Ir follows that for any $\delta > 0$, there is a constant C_δ such that

$$|\widehat{\mu}(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|), \qquad \zeta \in \mathbb{C},$$

where

$$H_K(\zeta) = \sup_{z \in K} \mathrm{Re}(z\zeta)$$

is the **support function** of K. H_K is a convex, positively homogeneous of $\zeta \in \mathbb{C} \cong \mathbb{R}^2$. In particular, $\widehat{\mu}$ is entire of order 1 and of **exponential type**:

$$|\widehat{\mu}(\zeta)| \leq Ce^{a|\zeta|}.$$

Proposition 1.1. *Let K be compact and convex with the support function H_K. Then $K = \{z \in \mathbb{C} : \mathrm{Re}(z\zeta) \leq H_K(\zeta)\, \forall \zeta \in \mathbb{C}\}$.*

Proof. ($\subseteq$): This inclusion is by definition of H_K.

($\supseteq$): Let $z_0 \notin K$. By the geometric Hahn-Banach theorem, there exists a hyperplane separating K and z_0. That is, there exists a real, linear form f on $\mathbb{R}^2$ and $\gamma \in \mathbb{R}$ such that $f(z) < \gamma < f(z_0)$ for any $z \in K$. There is a $\zeta \in \mathbb{C}$ such that $f(z) = \mathrm{Re}(z\zeta)$, so $H_K(\zeta) < \mathrm{Re}(z_0\zeta)$. $\square$

To summarize, if μ is carried by a compact K, then its transform $\mathcal{M}(\zeta) = \widehat{\mu}(\zeta)$ is entire and satisfies: for all $\delta > 0$, there exists a C_δ such that

$$|\mathcal{M}(\zeta)| \le C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

1.2 Inversion of the Fourier-Laplace transform

Theorem 1.1 (Polya, Ehrenpreis, Martineau[1])**.** *Let $K \subseteq \mathbb{C}$ be compact and convex, and let $\mathcal{M} \in \mathrm{Hol}(\mathbb{C})$ be such that*

$$|\mathcal{M}(\zeta)| \le C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

Then there exists a unique analytic functional μ such that $\widehat{\mu} = \mathcal{M}$ and μ is carried by K.

Proof. Idea: Construct the analytic functional μ using the **Borel transform** of $\mathcal{M}$. In particular, the estimate on $\mathcal{M}$ gives

$$|\mathcal{M}(\zeta)| \le C_1 e^{C|\zeta|}$$

for some C_1, C. When $R > 0$, we have

$$\frac{\mathcal{M}^{(j)}(0)}{j!} = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\mathcal{M}(\zeta)}{\zeta^{j+1}}\, d\zeta,$$

which gives

$$|M^{(j)}| \le j! C_1 e^{CR} R^{-1}.$$

The optimal choice of R is given by $R = j/C$. So we get

$$|\mathcal{M}^{(j)}(0)| \le j! C_1 e^j \left(\frac{C}{j}\right)^j \le C_1 (Ce)^j, \qquad j = 0, 1, 2, \ldots.$$

Define

$$B(\zeta) = \sum_{j=0}^{\infty} \zeta^{-j-1} \mathcal{M}^{(j)}(0).$$

[1] Polya proved the theorem in complex dimension 1. Ehrenpreis and Martineau generalized it to $\mathbb{C}^n$ for $n > 1$.

Then $B \in \mathrm{Hol}(\hat{\mathbb{C}} \setminus \{|\zeta| \leq Ce\})$, and $B(\infty) = 0$. Then function B is called the Borel transform of $\mathcal{M}$.

Let $\chi \in C_0^\infty(\mathbb{C})$ be such that $\chi = 1$ on a large disc, and define

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) f(\zeta) B(\zeta)\, d\lambda(\zeta),$$

where λ is Lebesgue measure in $\mathbb{C}$. Then μ is an analytic functional which is independent of the choice of χ. We claim first that $\widehat{\mu} = \mathcal{M}$: compute

$$\begin{aligned}
\widehat{\mu}^{(j)}(0) &= \mu(\zeta^j) \\
&= -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \zeta^j B(\zeta)\, d\lambda(\zeta) \\
&= \sum_{k=0}^{\infty} -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \zeta^j \zeta^{-k-1} \mathcal{M}^{(k)}(0)\, d\lambda(\zeta)
\end{aligned}$$

When $k = j$, the summand is

$$\mathcal{M}^{(j)}(0) \underbrace{\left(-\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta}\, d\lambda(\zeta) \right)}_{=1}$$

by the Cauchy integral formula. When $j \neq k$, it equals

$$\iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \zeta^\nu\, d\lambda(\zeta),$$

where $\nu \neq -1$. We can choose $\chi(\zeta) = \psi(|\zeta|^2)$ (making it radially symmetric to get:

$$\iint \psi'(|\zeta|^2) \zeta^{\nu+1}\, d\lambda(\zeta) = \iint \psi'(|\zeta|^2) r^{\nu+1} e^{i\theta(\nu+1)} r\, dr\, d\theta = 0.$$

We get $\widehat{\mu}^{(j)}(0) = \mathcal{M}^{(j)}(0)$. So $\widehat{\mu} = \mathcal{M}$ as their Taylor expansions agree.

We claim that B can be continued analyrically to $\hat{\mathbb{C}} \setminus K$. We will do this next time. $\square$